A NEW APPROACH TO DISPERSION ANALYSIS OF GRADED INDEX OPTICAL FIBERS

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ABSTRACT

The dispersion characteristics of graded index optical fiber are studied using a transverse transmission line circuit representation. For round fibers new approximations have been developed which significantly simplifies the analysis. An interesting perspective as to the physical mechanisms underlying the dispersion characteristics of such fibers result from circuit theoretic considerations.

1. Introduction

The two geometries considered are rectangular and circular. Such fibers consist of a core region surrounded by a cladding. Since the refractive index of the cladding is much less than that of the core, electromagnetic energy in this jacketed region will decay exponentially. It will be assumed that this cladding extends to infinity.

The approach taken here is to represent the continuously varying dielectric profile with a stepwise approximation. This is physically analogous to an optical waveguide consisting of stratified layers of dielectric material of different permittivities. Such a stepwise approximation facilitates an interesting circuit representation, in that an element of the strata is representable by a lossless transmission line whose characteristic impedance is frequency dependent. The transverse plane of the graded index fiber is thus represented by a cascade of these transmission lines with appropriate terminations. Since the type of profiles considered in this study decrease monotonically with increasing transverse distance, the transverse wavenumber can become imaginary. The transverse transmission line system thus in fact emulates the performance of the actual fiber by exhibiting a terminal plane on one side of which the lines propagate and beyond which they are cutoff. The location of this terminal plane depends on the spatial distribution of dielectric constants, the free space wave number $K_0 = \omega/c$, and the mode in question. It is this transverse cutoff which gives the graded index optical fiber its desirable dispersive properties. Evanescent sections of line that exist between propagating lines and the cladding provide shielding which in effect prevents the propagating sections of line from "seeing" the abrupt dielectric discontinuity at the core-clad interface. The propagating portion of the line system on the otherhand functions like an impedance matching transformer.

Unlike the slab which can be modeled with uniform transmission lines, the graded index rod necessitates the use of radial transmission lines. Several simplifying approximations for radial functions are obtained which permit a uniform line representation. These approximations are general in scope and suggest the global applicability of the concepts developed.

2. The Graded Index Dielectric Slab Waveguide

One of the simplest types of graded index optical waveguide is the dielectric sheet of uniform thickness and of infinite extent in the y-z plane. Figure 1 illustrates the transversly stratified approximation used to obtain a transmission line model.

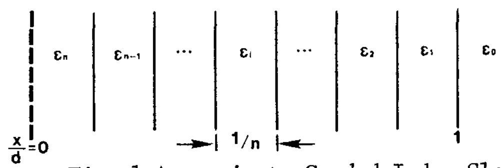

Fig. 1 Approximate Graded Index Slab

The procedure described below will concentrate only on the discrete quided modes of the graded index slab. The description is simplified by assuming that no component of electric or magnetic field contains y variation.

Only the transverse electric or TE modes, characterized by $E_z = 0$ are considered. The transverse magnetic or TM modes can be similarly obtained but are not given here.

Consider an arbitrary section of the stratified approximation of figure 1. This section is of length $(x_0 - x)$ and uniform permittivity ϵ_i. The transverse transmission line charactristics are obtained from Maxwell's equations. For the steady state angular frequency ω and z-direction variation $\exp(j\beta z)$, the resulting open circuit impedance matrix of the 2-port representing a width of transverse slab from x to x_0 is given by

$$Z_{TE} = -jZ \begin{bmatrix} \operatorname{ctn} k(x_o - x) & \csc k(x_o - x) \\ \csc k(x_o - x) & \operatorname{ctn} k(x_o - x) \end{bmatrix} \qquad (1)$$

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where $Z = K_0Z_0/k$, $k = \sqrt{\epsilon K_0^2 - \beta^2}$, $\epsilon = \epsilon_i/\epsilon_0$

This transmission line representation has been obtained on the assumption that the line is propagating in the x-direction, but the behavior of the graded index waveguide is intimately related to the existance of a transverse plane beyond which the slab is transversly cutoff. Such a section of dielectric slab is represented by a cutoff line whose corresponding 2-port impedance matrix is obtained by setting $k=-jh$ in eq. (1).

With the intervening elements of the strata each specified, it now becomes necessary to specify the terminations. The cladding is represented by an infinite cutoff line whose input impedance is obtained from equation (1) by letting x_0 approach infinity with x fixed. This gives $Z=jK_0Z_0/h$. A lumped impedance of this value can then be used to represent the cladding. The inner most termination depends upon the class of TE modes. For symmetric the E_y is maximum at x=0 which corresponds to an open circuit termination.

The value of forward propagation constant is obtained using transverse resonance techniques [4].

For the numerical results the innermost or core section has been assigned a relative permittivity of 2.45. The outermost section or cladding has a relative permittivity of unity. For square law indexing the intervening sections have relative permittivity given by

$$\frac{\epsilon i}{\epsilon_0} = \epsilon_{core} - \left[\frac{n+1-i}{n-i}\right]^2 ; \; i = 2,3,\ldots,n \qquad (2)$$

where n is the total number of sections, Δ is a parameter chosen to obtain the desired minimum permittivity for the penultimate section. The group delay is computed by means of a five point polynomial approximation for $d\beta/dK_0$. The tabular results are given in terms of a "dispersion index", $\hat{\tau}_g$ defined by

$$\hat{\tau}_g = d\beta/dK_0 - \sqrt{\varepsilon core} \qquad (3)$$

Kirchhoff [2] has solved the exact differential equation for the case of a $1-x^2$ type dielectric profile. These results are used for direct comparison.
Table 1 compares the exact propagation constant and dispersion index of the graded index slab with a 20 line approximation.

3. The Graded Index Dielectric Rod Waveguide

The structure to be studied is shown in figure 2. The solutions to Maxwell's equtions will be a travelling wave of the form

$$\underline{E} = \underline{E}(r)e^{j(n\phi - \beta z)}, \; \underline{H} = \underline{H}(r)e^{j(n\phi - \beta z)} \qquad (4)$$

The TE and TM mode designation exists only for n=0. For n>0 all six electromagnetic field components are necessary to describe the discrete guided modes (termed hybrid modes) and are designated as HE or EH.

Consider one element of the stratified waveguide of figure 2 which can be envisioned as an annular dielectric ring of uniform permittivity.

GRADED INDEX SLAB
TE_{on} Symmetric Modes
$K_0 = 40 \quad \Delta = 0.05$

Propagation Constant β

n	exact	20-line approx.
1	62.54	62.54
2	62.25	62.25
3	61.91	61.90
4	61.45	61.44
5	60.83	60.82

Dispersion Index $\hat{\tau}_g$

n	exact	20-line approx.
1	9.65×10^{-6}	7.79×10^{-5}
2	7.51×10^{-4}	8.57×10^{-4}
3	5.90×10^{-3}	5.8×10^{-3}
4	1.69×10^{-2}	1.67×10^{-2}
5	3.25×10^{-2}	3.23×10^{-2}

$\frac{10}{3.}\hat{\tau}_g$ has units of micro seconds/kilometer

TABLE 1

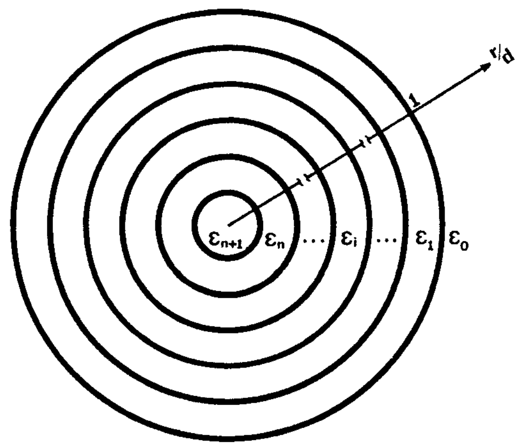

Fig. 2 Approximate Graded Index Rod

Consider the TE modes. The resulting 2-port impedance matrixis obtained from Maxwell's equations and is given by

$$Z_{TE} = j\begin{bmatrix} Z(r)\frac{M'}{M} & \frac{2}{y}\frac{Z(r_0)}{M} \\ \frac{2}{y}\frac{Z(r_0)}{M} & -Z(r_0)\frac{M'}{M} \end{bmatrix} \qquad (5)$$

where Z(r) is the radial characteristic impedance

$$Z(r) = 2\pi r K_0 Z_0/k \qquad (6)$$

The terms M and M' are radial functions

$$M(x,y) = J_n(x)N_n(y) - J_n(y)N_n(x) \qquad (7)$$

$$M'(x,y) = J'_n(x)N_n(y) - J_n(y)N'_n(x) \qquad (8)$$

$x=kr$, $y=kr_0$, J_n and N_n are Bessel functions of the first and second kind (n=0 for TE modes)

For the HE_{11} mode azimuthal variation of field components make the formulation more complex. Simplification occurs by noting that for this lowest order HE mode the z-component of the electric field is negligibly small [6]. The approximation $E_z=0$ results in a single radial line representation whose 2-port impedance matrix is

$$Z_{HE} = -j \begin{bmatrix} Z(r)\frac{M'}{M} & \frac{2Z(r)}{y}\frac{1}{M} \\ \frac{2Z(r_o)}{y}\frac{1}{M} & -Z(r_o)\frac{M'}{M} \end{bmatrix} \quad (9)$$

Radial transmission lines differ from uniform transmission lines in two major respects. The first is the lack of a unique characteristic impedance. Secondly, the transverse fields are described with radial functions.

The simplest radial functions, occur when n = 0 [7] and are given by

$$Cs(x,y) = \pi y \left\{J_1(y)N_o(x) - J_o(x)N_1(y)\right\} /2$$
$$cs(x,y) = \pi y \left\{J_1(x)N_o(y) - J_o(y)N_1(x)\right\} /2$$
$$Sn(x,y) = \pi y \left\{J_1(y)N_1(x) - J_1(x)N_1(y)\right\} /2$$
$$sn(x,y) = \pi y \left\{J_0(y)N_o(x) - J_o(x)N_o(y)\right\} /2 \quad (10)$$

The functions cs and Cs are termed the small and large radial cosine, and sn and Sn are the small and large radial sine functions respectively. Define the small and large radial cotangent as

$$ct(x,y) = \frac{cs(x,y)}{-sn(x,y)}, \quad Ct(x,y) = \frac{Cs(x,y)}{-Sn(x,y)} \quad (11)$$

These radial cotangents satisfy

$$ct(x,y)\,\rho(x,y) = -Ct(x,y) \quad (12)$$

where $\rho(x,y) = sn(x,y)/Sn(x,y)$

Approximations to the radial functions are given by

$$Ct(x,y) \cong \frac{x\,\ln(y/x)}{(y-x)}\,ctn(y-x) \quad (13)$$

$$ct(x,y) \cong \frac{(y-x)}{x}\frac{ctn(y-x)}{\ln(y/x)} \quad (14)$$

$$\rho(x,y) = -\frac{Ct(x,y)}{ct(x,y)} \cong \frac{xy\,\ln^2(y/x)}{(y-x)^2} \quad (15)$$

The radial cosecant (cst) is computed with the identity

$$cst^2(x,y) = \left\{1 + ct(x,y)\,Ct(y,x)\right\} / \rho(x,y) \quad (16)$$

Equations (13) through (15) were obtained in a semi-emperical manner employing numerical curve fitting.

Figures 3 and 4 show the relative error for the radial cotangent ct(x,y) and radial cosecant cst(x,y) and their approximations, respectively, for electrical lengths as fractions of normalized unit to total radius, with $y-x = k(r_0-r)$, $r<r_0<1$ with $r_0-r=1/25$ and $k = 1,2,...,20$

The approximations are no longer applicable for values of r approaching zero. This is not the serious concern since the innermost line is readily dealt with in an exact manner.

These approximations may now be substituted into the impedance matrices previously obtained. For TE modes, Z_{TE} becomes

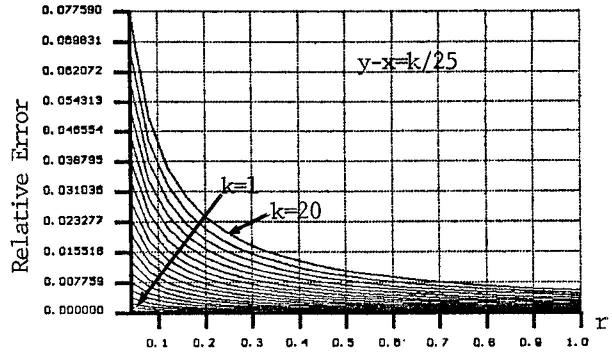

Fig. 3 Relative Error ; ct(x,y)

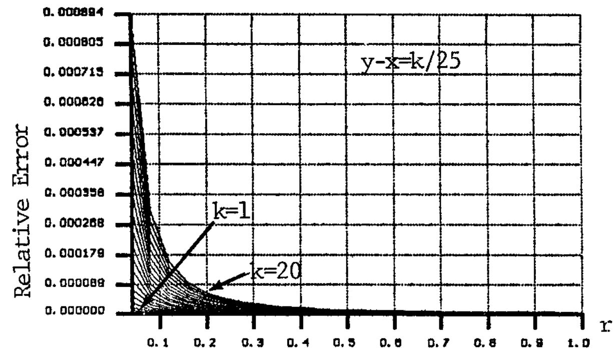

Fig. 4 Relative Error ; cst(x,y)

$$Z_{TE} = -jZ(r,r_o) \begin{bmatrix} ctn\,k(r_o-r) & csc\,k(r_o-r) \\ csc\,k(r_o-r) & ctn\,k(r_o-r) \end{bmatrix} \quad (17)$$

$$\text{where } Z(r,r_o) = \frac{2\pi}{k} K_o Z_o \frac{(r_o-r)}{\ln(r_o/r)} \quad (18)$$

The interesting consequence of the approximation is to yield a uniform transmission line of electrical length $k(r_0-r)$ and characteristic impedance $Z(r,r_0)$. The resulting description, once the parameters r and r_0 have been specified, differ from the results obtained for the dielectric slab only in the definition of the characteristic impedance.

A similar procedure is followed for the HE_{11} mode. The radial functions are defined by eq. (7) and (8) with n=1.

The impedance matrix for the HE_{11} mode (with $E_z=0$) is given by

$$Z = -jZ(r,r_o) \begin{bmatrix} r^2 ctn\,u & rr_o csc\,u \\ rr_o csc\,u & r_o^2\,ctn\,u \end{bmatrix} - jZ_s \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (19)$$

$$Z(r,r_o) = kZ_s \frac{\ln(r_o-r)}{(r_o-r)}, \quad Z_s = \frac{2\pi}{k^2} K_o Z_o, \quad u = k(r_o-r) \quad (20)$$

Once again we have a radial line model in which the incremental section employs a uniform line. This uniform transmission line representation is best visualized by means of the circuit model shown in figure 5.

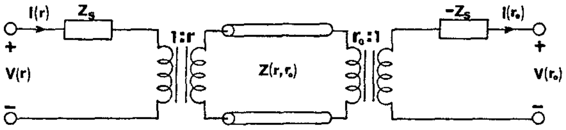

Fig. 5 Uniform Transmission Line Model

Again transversly evanescent lines can be obtained by substituting k=jh.

The core and cladding terminations still must be specified. The radial function approximations are not valid for a section of line which includes r = 0. This problem is circumvented by moving one section to the right and computing the core impedance from equation(17) for TE modes or equation (19) for the HE_{11} mode by taking the limit as $x \to 0$, y fixed, yielding

$$Z_{core} = \begin{cases} -jZ(r_o)J_o'(y)/J_o(y) & \text{TE modes} \\ -jZ(r_o)J_1'(y)/J_1'(y) & HE_{11} \text{ mode} \end{cases} \qquad (21)$$

The cladding is defined by a cutoff line of infinite length. Thus, for x fixed and $y \to \infty$

$$Z_{clad} = \begin{cases} -jZ(r)K_o'(x)/K_o(x) & \text{TE modes} \\ -jZ(r)K_1'(x)/K_1(x) & HE_{11} \text{ mode} \end{cases} \qquad (22)$$

K_n is the modified Bessel function of order n. All comments pertaining to the numerical results for the slab carry over here. The ability of the uniform line cascade to accurately model a radial transmission line can be verified by considering a uniform (no grading) dielectric rod since exact results for this case are obtained readily. Such a comparison for the HE_{11} mode is shown in table 2. Tables 3 and 4 give the dispersion index for the TE_{on} and HE_{11} modes of the uniform line model of the graded index rod with a square law profile. Note that a significantly larger number of lines is necessary for the HE_{11} mode because of the additional approximations made.

UNIFORM DIELECTRIC ROD

HE_{11} mode, $K_o = 50$

β	$\hat{\tau}g$	no. of lines
78.23	$7.06x10^{-4}$	exact
78.22	$7.95x10^{-4}$	30
78.22	$7.54x10^{-4}$	40
78.23	$7.27x10^{-4}$	50

TABLE 2

GRADED INDEX ROD

TE_{on} modes, $K_o = 50$, $\Delta = 0.05$

n	$\hat{\tau}g$ (40-lines)	$\hat{\tau}g$ (50-lines)
1	$1.43x10^{-4}$	$1.25x10^{-4}$
2	$1.36x10^{-3}$	$1.35x10^{-3}$
3	$6.5\ x10^{-3}$	$6.51x10^{-3}$
4	$1.56x10^{-2}$	$1.57x10^{-2}$
5	$2.76x10^{-2}$	$2.77x10^{-2}$

TABLE 3

GRADED INDEX ROD

HE_{11} mode, $K_o = 50$, $\Delta = 0.05$

No. of lines	$\hat{\tau}g$
100	$1.91x10^{-5}$
125	$1.85x10^{-5}$
150	$1.77x10^{-5}$
175	$1.68x10^{-5}$
200	$1.62x10^{-5}$

TABLE 4

$\frac{10}{3}\ \hat{\tau}g$ has units of microseconds/kilometer

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